MATH 2230 Complex Variables with Applications (2014-2015, Term 1) Suggested Solution to HW2

1. (SEC.18,No.5)

   Proof: When $z = (x, 0)$ is a nonzero point on the real axis,

   $$f(z) = (\frac{x + i0}{x - i0})^2 = 1.$$

   When $z = (0, y)$ is a nonzero point on the imaginary axis,

   $$f(z) = (\frac{0 + iy}{0 - iy})^2 = 1.$$

   When $z = (x, x)$ is a nonzero point on the line $y = x$,

   $$f(z) = (\frac{x + ix}{x - ix})^2 = (\frac{1 + i}{1 - i})^2 = -1.$$

   Thus, the limit of $f(z)$ as z tends to 0 does not exist.

2. (SEC.18,No.10)

   Solution: (a) By theorem in Sec.17, we have

   $$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4 \textit{ since } \lim_{z\to 0} \frac{4(\frac{1}{z})^2}{(\frac{1}{z} - 1)^2} = \lim_{z\to 0} \frac{4}{(1-z)^2} = 4.$$

   (b) By theorem in Sec.17, we have

   $$\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty \textit{ since } \lim_{z\to 1}(z-1)^3 = 0.$$

   (c) By theorem in Sec.17, we have

   $$\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty \textit{ since } \lim_{z\to 0} \frac{\frac{1}{z} - 1}{(\frac{1}{z})^2 + 1} = \lim_{z\to 0} \frac{z - z^2}{1 + z^2} = 0.$$

3. (SEC.18,No.11)

   Proof: (a) Since

$$\lim_{z\to\infty}\frac{1}{T(\frac{1}{z})}=\lim_{z\to 0}\frac{\frac{c}{z}+d}{\frac{a}{z}+b}=\lim_{z\to 0}\frac{c+dz}{a+bz}=0\ (a\neq 0\ since\ \ ad-bc\neq 0\ and\ c=0)$$

   We have

$$\lim_{z\to\infty}T(z)=\infty$$

   (b) Since

$$\lim_{z\to 0}T(\frac{1}{z})=\lim_{z\to 0}\frac{\frac{a}{z}+b}{\frac{c}{z}+d}=\lim_{z\to 0}\frac{a+bz}{c+dz}=\frac{a}{c}\ (since\ c\neq 0)$$

   We have

$$\lim_{z\to\infty}T(z)=\frac{a}{c}$$

   And since

$$\lim_{z\to -\frac{d}{c}}\frac{1}{T(z)}=\lim_{z\to -\frac{d}{c}}\frac{cz+d}{az+b}=0\ (since\ ad-bc\neq 0)$$

   We have

$$\lim_{z\to -\frac{d}{c}}T(z)=\infty$$

4. (SEC.20,No.8)
   Proof: Refer to Page 56 on the textbook.

5. (SEC.20,No.9)
   Proof:

$$\frac{\triangle w}{\triangle z}=(\frac{\overline{\triangle z}}{\triangle z})^2$$

   If $\triangle z=(\triangle x,0)$, then

$$\frac{\triangle w}{\triangle z}=(\frac{\triangle x}{\triangle x})^2=1$$

   If $\triangle z=(0,\triangle y)$, then

$$\frac{\triangle w}{\triangle z}=(\frac{-i\triangle y}{i\triangle y})^2=1$$

   If $\triangle z=(\triangle x,\triangle x)$, then

$$\frac{\triangle w}{\triangle z}=(\frac{\triangle x-i\triangle x}{\triangle x+i\triangle x})^2=-1$$

   Thus, $f'(0)$ does not exist.

6. (SEC.24,No.2)
   Remark: Refer to Sec.23 on the textbook.